Problem 5291. Let m_a, m_b be the medians of a triangle with side lengths a, b, c. Prove that:

$$m_a m_b \leq \frac{2c^2 + ab}{4}$$

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By using the well known formulas

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}, \qquad m_b = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2}$$

the given inequality, after some algebra, rewrites as

$$\begin{aligned} \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \cdot \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2} \leq \frac{2c^2 + ab}{4} \qquad &\Leftrightarrow \\ \left(2b^2 + 2c^2 - a^2\right)\left(2a^2 + 2c^2 - b^2\right) \leq \left(2c^2 + ab\right)^2 \qquad &\Leftrightarrow \\ 2(a-b)^2(a+b-c)(a+b+c) \geq 0 & \end{aligned}$$

which is true. The equality holds if and only if $a = b$ □